ON THE MOTION OF A GYROSCOPE SUPPORTED BY BALL BEARINGS ON GIMBALS

(O DVIZHENII GIROSKOPA, USTANOVLENNOGO NA SHARIKOVYKH PODSHIPNIKAKH V KARDANOVOM PODVESE)

PMM Vol.26, No.2, 1962, pp. 365-369

S. A. KHARLAMOV
(Moscow)

(Received July 10, 1961)

In the theory of a free gyroscope, it is assumed that the location of its instantaneous axis of rotation relative to the inner ring of the gimbals is fixed in the coordinate system attached to this ring. Below is considered the motion of a gyroscope on gimbals for which the stated assumption is not made. The relative motion of the gyroscope is determined purely kinematically by means of prescribing the motion of its center of inertia and the orientation of the figure axis. The inertia forces of the relative motion generate small oscillations of the gyrsocope in the neighborhood of the stationary motion causing it to precess relative to the axis of the outer ring. Such a relative motion can take place if the gyroscope is supported by ball bearings on gimbals.

1. The motion of a gyroscope relative to the inner ring of the gimbals. The axis of the gyroscope figure describes in relative motion a one-sheeted hyperboloid of rotation, the equation of which is derived below. The axis of the hyperboloid passes through the suspension center and is perpendicular to the axis of rotation of the inner ring.

Let us attach to the inner ring of the gimbals a system of coordinates $Oxyz$ such that the axis Oy is directed along the axis of rotation of the inner ring relative to the outer ring, while the axis Oz is along the axis of the hyperboloid.

The origin of the C system of coordinates $C\xi\eta\zeta$ attached to the gyroscope will be fixed at the center of inertia, while the axis $C\zeta$ will be directed along the figure axis. The location of the coordinate system $C\xi\eta\zeta$ relative to the system $Oxyz$ will be defined by three coordinates of point C and by the three Euler angles ψ, θ, φ (Fig. 1).

The prescribed motion of the gyroscope relative to the inner ring of the suspension will be given in the form

$$x_c = h\cos\psi, \qquad y_c = h\sin\psi, \qquad z_c = 0$$

$$\psi = \psi(t), \qquad \theta = \theta_0, \qquad \varphi = \frac{k-1}{\cos\theta_0}\,\psi(t) + \varphi_0$$

The gyroscope location is uniquely defined by a single coordinate.

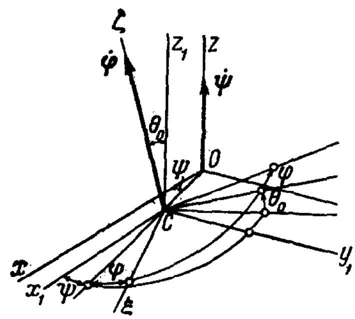

Fig. 1.

Let us now define the projections of the relative angular velocity of the gyroscope and the relative velocity of the point C on the axis of the $Oxyz$ system

$$\omega_x = (k-1)\tan\theta_0\dot\psi\sin\psi, \qquad \omega_y = -(k-1)\tan\theta_0\dot\psi\cos\psi \qquad \omega_z = k\dot\psi$$

$$v_{cx}^{(r)} = -h\dot\psi\sin\psi, \qquad v_{cy}^{(r)} = h\dot\psi\cos\psi, \qquad v_{cz}^{(r)} = 0$$

The geometric interpretation of motion is considered in the book by Suslov.*

We will choose two points P and Q on the axis $C\zeta$ the coordinates of which $(0,\ 0,\ l)$ and $(0,\ 0,\ -l)$ will be called the ends of the figure axis. Their trajectories in the $Oxyz$ coordinate system represent the circles

$$x^2 + y^2 = h^2 + l^2\sin^2\theta_0, \qquad z = \pm l\cos\theta_0$$

Eliminating l we will find the equation of the hyperboloid of rotation, described by the figure axis in the process of rotation

$$x^2 + y^2 - z^2\tan^2\theta_0 = h^2$$

We will consider the motion of a gyroscope in which the figure axis coincides with the polar axis of inertia; and at the same time we will assume the smallness of the angle θ_0 and let $\sin\theta_0 = \theta_0$ and $\cos\theta_0 = 1$.

2. The equations of motion of the gyroscope. We will derive by the Lagrange method the equations of motion for the gyroscope fixed on an immovable support relative to the inertial space. The kinetic energy of the gimbals consists of the kinetic energy of the outer ring $T_2 = 1/2A_2\dot\alpha$, (where A_2 is the moment of inertia of the outer ring relative to its axis of rotation, α is the angle of rotation of the ring relative to the base) and of the kinetic energy of the inner ring

---

* See [1] example 25 on pp. 96-97 and example 32 on pp. 110-111.

$$T_1 = \frac{1}{2}(I_{xx}p_1^2 + I_{yy}q_1^2 + I_{zz}r_1^2 - 2I_{xy}p_1q_1 - 2I_{yz}q_1r_1 - 2I_{xz}p_1r_1)$$

Here I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{yz}, I_{xz} are moments of inertia of the inner ring relative to the system $Oxyz$; p_1, q_1, r_1 are projections of the absolute angular velocity of the inner ring on the axes of the $Oxyz$ coordinates attached to it. If β is used to denote the rotation angle of the inner ring relative to the outer ring, then

$$p_1 = \dot{\alpha}\cos\beta, \qquad q_1 = \dot{\beta}, \qquad r_1 = \dot{\alpha}\sin\beta$$

In the following we will consider that the axis of rotation of the inner ring relative to the outer ring is a principal axis of inertia, i.e. $I_{xx} = A_1$, $I_{yy} = B_1$, $I_{zz} = C_1$, $I_{xz} = E_1$, $I_{xy} = I_{yz} = 0$. In order to determine the kinetic energy of the gyroscope one computes the components of its inertia tensor relative to the system $Cx_1y_1z_1$, the origin of which is at the point C, while the axes are parallel to the axes Ox, Oy and Oz

$$\begin{aligned}
I_{x_1x_1} &= A + (C - A)\sin^2\psi\sin^2\theta_0, & I_{x_1y_1} &= (C - A)\cos\psi\sin\psi\sin^2\theta_0\\
I_{y_1y_1} &= A + (C - A)\cos^2\psi\sin^2\theta_0, & I_{x_1z_1} &= -(C - A)\sin\psi\cos\theta_0\sin\theta_0\\
I_{z_1z_1} &= C - (C - A)\sin^2\theta_0, & I_{y_1z_1} &= (C - A)\cos\psi\cos\theta_0\sin\theta_0
\end{aligned}$$

and then one finds the projections of the absolute angular velocity of the gyroscope and the projections of the absolute velocity of its center of inertia on the same axes.

The angular velocity of the inner ring is the translational angular velocity of the gyroscope, and since the relative angular velocity has been determined in Section 1, we obtain p', q', r', the projections of the absolute angular velocity on the axes Cx_1, Cy_1, Cz_1 respectively

$$p' = \dot{\alpha}\cos\beta + (k - 1)\tan\theta_0\dot{\psi}\sin\psi, \qquad q' = \dot{\beta} - (k - 1)\tan\theta_0\dot{\psi}\sin\psi, \quad r' = \dot{\alpha}\sin\beta + k\dot{\psi}$$

The projections of the absolute velocity of the gyroscope center of inertia on the $Oxyz$ coordinate axes is determined by the formulas

$$v_{cx} = -h(\dot{\psi} + \dot{\alpha}\sin\beta)\sin\psi, \qquad v_{cy} = h(\dot{\psi} + \dot{\alpha}\sin\beta)\cos\psi$$
$$v_{cz} = h(\dot{\alpha}\cos\beta\sin\psi + \dot{\beta}\cos\psi)$$

Adding the kinetic energy of the gyroscope to that of the suspension, we obtain the kinetic energy of the total system (m is the mass of the gyroscope).

$$\begin{aligned}
2T = {} & (A_2 + A_1\cos^2\beta + C_1\sin^2\beta)\dot{\alpha}^2 - 2E_1\dot{\alpha}\dot{\beta}\sin\beta\cos\beta + B_1\dot{\beta}^2 + I_{x_1x_1}p'^2 + I_{y_1y_1}q'^2 + \\
& + I_{z_1z_1}r'^2 - 2I_{x_1y_1}p'q' - 2I_{y_1z_1}q'r' - 2I_{x_1z_1}p'r' + \\
& + mh^2[(\dot{\psi} + \dot{\alpha}\sin\beta)^2 + (\dot{\alpha}\cos\beta\sin\psi + \dot{\beta}\cos\psi)^2]
\end{aligned}$$

Taking α, β and ψ as the generalized coordinates of the system, assuming that there is no friction in the axes and that the centers of inertia of the rings are coincident with the suspension center, we derive by known procedures the equations of motion for the system. The so-obtained equations are quite cumbersome but they can be simplified with good accuracy by taking into account the fact that the angle is very small, i.e. $\cos\theta_0 = 1$ and $\sin\theta_0 = \theta_0$. If the gyroscope rotates sufficiently rapidly and the angular velocity ψ is high then by assumption $\theta_0\dot{\psi} \sim \dot{\alpha} \sim \dot{\beta}$ and $\theta_0\dot{\psi}^2 \sim \ddot{\alpha} \sim \ddot{\beta}$. In the equations of motion of the gyroscope, we neglect the terms of order higher than $\theta_0{}^2\dot{\psi}^2$ assuming that $mh^2 \sim ml^2\theta_0{}^2$, while in practical gyroscopes $ml^2 \sim A$

$$[A_2 + (A + A_1)\cos^2\beta + C_1\sin^2\beta - 2E_1\sin\beta\cos\beta]\ddot{\alpha} -$$
$$-2[(A + A_1 - C_1)\sin\beta\cos\beta + E_1(\cos^2\beta - \sin^2\beta)]\dot{\alpha}\dot{\beta} + C(k\dot{\psi} + \dot{\alpha}\sin\beta)\dot{\beta}\cos\beta -$$
$$-(kC - A)\theta_0(\dot{\psi}^2\cos\psi\cos\beta - \dot{\psi}\dot{\beta}\sin\psi\sin\beta) = 0$$

$$(A + B_1)\ddot{\beta} + [(A + A_1 - C_1)\sin\beta\cos\beta + E_1(\cos^2\beta - \sin^2\beta)]\dot{\alpha}^2 -$$
$$-C(k\dot{\psi} + \dot{\alpha}\sin\beta)\dot{\alpha}\cos\beta - (kC - A)\theta_0(\dot{\psi}^2\sin\psi + \dot{\alpha}\dot{\psi}\sin\beta\cos\psi) = 0$$
$$C\frac{d}{dt}(k\dot{\psi} + \dot{\alpha}\sin\beta) + (kC - A)\theta_0(\ddot{\alpha}\cos\beta\sin\psi + \ddot{\beta}\cos\psi) = 0$$

3. Solution of the equations of motion for the gyroscope. If $\theta_0 = 0$, the equations of motion permit a particular solution

$$\alpha = \alpha_0 = \text{const}, \qquad \beta = \beta_0 = \text{const}, \qquad \psi = \lambda t + \psi_0$$

corresponding to the stationary motion of the gyroscope relative to immovable gimbals. At any instant of time the motion of the gyroscope represents a rotation relative to some axis with the absolute angular velocity $\omega = k\lambda$. The instantaneous axis of rotation translates in space describing a cylindrical surface the axis of which is Oz. Consequently, for $\theta_0 = 0$ the orientation of the axis Oz relative to the immovable base remains fixed.

We will consider terms containing $\theta_0 \neq 0$ as small perturbations, and will represent the solutions of equations in the form

$$\alpha = \alpha_0 + ut + \xi(t), \qquad \beta = \beta_0 + \eta(t), \qquad \psi = \lambda t + \psi_0 + \zeta(t)$$

where u is the constant precession velocity of the gyroscope representing a small quantity of order $\theta_0{}^2\dot{\psi}$; $\xi(t)$ and $\eta(t)$ are bounded functions of time of smallness θ_0. If one substitutes the derived equations for α and β into the equations of motion and retain only the terms of order $\theta_0\dot{\psi}^2$ then one obtains the following system of nonhomogeneous linear differential equations

$$[A_2 + (A + A_1)\cos^2\beta_0 + C_1\sin^2\beta_0 - 2E_1\sin\beta_0\cos\beta_0]\ddot{\xi} +$$
$$+ C\omega\cos\beta_0\dot{\eta} = (C\omega - A\lambda)\lambda\theta_0\cos\beta_0\cos(\lambda t + \psi_0)$$

$$(A+B_1)\ddot{\eta}-C\omega\cos\beta_0\dot{\xi}=(C\omega-A\lambda)\lambda\theta_0\sin(\lambda t+\psi_0), \qquad k\ddot{\zeta}+\ddot{\xi}\sin\beta_0=0$$

The forcing moments on the right-hand side of the equations can be considered as projections on the axes of the outer and inner rings of the constant moment directed along OC representing the sum of the moment of centrifugal forces $(C-A)\lambda^2\theta_0$ and some gyroscopic* moment $C(\omega-\lambda)\lambda\theta_0$.

The first two equations of the system are integrated independently of the third

$$\xi=-(kC-A)\theta_0\cos\beta_0\frac{kC+I_1}{I_0I_1-k^2C^2\cos^2\beta_0}\cos(\lambda t+\psi_0)$$

$$\eta=-(kC-A)\theta_0\frac{I_0+kC\cos^2\beta_0}{I_0I_1-k^2C^2\cos\beta_0}\sin(\lambda t+\psi_0)$$

where

$$I_0=A_2+(A+A_1)\cos^2\beta_0+C_1\sin^2\beta_0-2E_1\cos\beta_0\sin\beta_0, \qquad I_1=A+B_1$$

The angular velocity of precession u is found by the Magnus [2] method according to which the obtained solutions for ξ and η should be substituted into the equations of motion and the result averaged over the period of oscillation $2\pi/\lambda$. As a result we obtain the following equation

$$-C\omega\cos\beta_0 u=a^2\lambda^2[(A_2+C_1)\operatorname{tg}\beta_0-E_1]$$

where a is the amplitude of small oscillations of the gimbals' outer ring. From this we derive a formula for the drift velocity of the gyroscope under the action of nutational oscillations

$$u=-\theta_0^2\lambda\frac{[(A_2+C_1)\sin\beta_0-E_1\cos\beta_0](kC-A)^2(kC+I_1)^2}{2kC(I_0I_1-k^2C^2\cos^2\beta_0)^2}$$

The result obtained indicates the necessity for accurate coincidence of the Oz axis with the principal axis of inertia of the outer ring.

4. Determination of bearing reactions on gimbals. We will now show that the displacement h of the gyroscope center of inertia causes additional reactions in the gyroscope bearings of the gimbals. The generalized reaction forces will be determined by the method of Lur'e [3]. Let the gyroscope be mounted such that the outer ring axis is horizontal, while the axis Oy is directed along the gravity vertical (Fig. 2). The

---

* In the system of coordinates, the gyroscope rotating relative to the axis Oz with angular velocity λ has a certain kinetic moment H' directed along $C\zeta$. The axes Oz and $C\zeta$ do not coincide, therefore, a gyroscopic moment $H'\times\lambda$ equal to $C(\omega-\lambda)\lambda\theta_0$ directed along OC (Fig. 1) is generated.

vertical reaction of the thrust bearing is to be determined.

Let us eliminate the constraint by removing the support of the inner ring and write down the kinetic energy for the inner ring and the gyroscope

$$T = \frac{1}{2}\left[M_1\dot{y}^2 + m\left(\dot{y}^2 - 2h\dot{y}\dot{\psi}\cos\psi + h^2\dot{\psi}^2\right) + k^2C\dot{\psi}^2\right]$$

where y is the coordinate of the center of inertia of the inner ring M_1 and m are masses of the inner ring and of the gyroscope. In computing the kinetic energy we let $\theta_0 = 0$ and neglect the kinetic energy due to precession resulting from the earth's rotation. The potential energy is of the form

$$U = (M_1 + m)\,gy + mgh\sin\psi$$

Writing the Lagrange equations and regarding the vertical component of the thrust bearing as the generalized reaction force we obtain

$$(M_1 + m)\,\ddot{y} + mh\left(\ddot{\psi}\cos\psi - \dot{\psi}^2\sin\psi\right) = -(M_1 + m)\,h + Y_1$$

$$(k^2C + mh^2)\,\ddot{\psi} + mh\,(\ddot{y} + g)\cos\psi = 0$$

If the constraint equation $y = 0$ is considered, then the latter equation is integrated independently from the first one, while the thrust bearing reaction is found from the first equation.

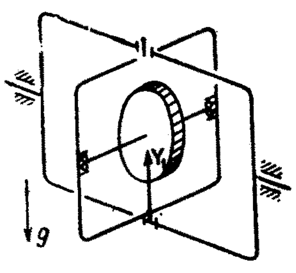

Fig. 2.

Since h is quite small we take only the zeroth approximation for the second equation

$$\psi = \lambda t + \psi_0$$

From the first equation we find that

$$Y_1 = (M_1 + m)\,g - mh\lambda^2\sin(\lambda t + \psi_0)$$

In the case when the constrain is not rigid, i.e.

$$y \geqslant 0, \text{ when } h\lambda^2 > \frac{M_1 + m}{m}\,g$$

the contact in the thrust bearing can be broken.

The reactions in the supports of the gyroscope and the inner ring can be found by analogous means.

5. Displacement of the gyroscope figure axis in the presence of ball bearing rotation. Ball bearings are usually used for support of the gyroscope in the inner ring of the gimbals. Reactions acting on the inner races of the bearings result from contact deformations of the balls and races. If the balls are not uniform in size and stiffness then the axis of the inner race of the bearing may not coincide with the axis of the outer race. Let, for example, the outer race of the bearing be coaxial and ideally symmetric relative to the mutual axis and the inner races be so located that the figure axis of the gyroscope $C\zeta$ is the axis of symmetry of these races. Then the displacement of the figure axis relative to the axis of symmetry of the outer races remains fixed in the coordinate system rotating with the separator since the distribution of the balls and their deformations depend on the rotation angle of the separator. The axis of symmetry of the outer races will be represented by the axis Oz.

In the absence of slipping between the balls and the races of the bearing, the angular velocities of the gyroscope and those of the separators are proportional.

Let the displacements of the ends of the figure axis be identical; then the considered case of relative motion of the gyroscope applies, if only the ball bearing separators rotate in synchronism. The angle ψ is the rotation angle for the separator about the axis Oz, while the angular velocity of the separator λ is determined from the angular velocity of rotation ω according to the known [4] formula

$$\lambda = \frac{\omega}{k} = \frac{D_0 - d\cos\delta}{2D_0}\,\omega$$

where D_0 is the diameter of the circle passing through the ball centers, d is the ball diameter, and δ is the contact angle. It follows from this formula that $k > 2$.

If the radius R of a circle described by the figure axis is known, then the possible values of the angle θ_0 and displacement h are bounded in the following regions

$$0 \leqslant \theta_0 \leqslant \frac{R}{l}, \qquad 0 \leqslant h \leqslant R$$

The analysis of the formula for the gyroscope drift velocity in the presence of nutational oscillations shows that the motion of a gyroscope supported by ball bearings with displacement of the figure axis relative to the symmetry axis of the outer bearing races is analogous to the motion of a dynamically unbalanced gyroscope in which the polar moment of inertia is k times larger than $C^* = kC$, the angle between the axis of dynamic symmetry and the axis of rotation equals θ_0, and the velocity of rotation is k times smaller and equal to λ. The drift velocity vanishes for $A = kC$, i.e. for gyroscopes with highly elongated ellipsoid of inertia since $k > 2$. Usually the gyroscopes have a flattened ellipsoid of inertia, therefore, in the presence of bearing rotation the influence of gyroscope axis of symmetry displacement upon drift can be of the same order of magnitude as the influence of dynamic unbalance investigated by Klimov [5].

It is assumed that in the scheme investigated the bearing separators are rotating synchronously. Under actual conditions strict synchronism is probably not observed; therefore, it would be interesting to investigate the case when the ends of the gyroscope figure axis rotate with different angular velocities relative to the axis Oz.

Also, under actual conditions it is necessary to investigate more thoroughly the displacement of the gyroscope figure axis in the process of motion. It is known that an end of a shaft rotating in a ball bearing describes a complicated trajectory. An example of such a trajectory is given by Yamamoto [6] who considers oscillations of a disc and an elastic shaft rotating in ball bearings.

BIBLIOGRAPHY

1. Suslov, G.K., *Teoreticheskaia mekhanika* (*Theoretical Mechanics*). 3rd Ed., Gostekhizdat, 1946.

2. Magnus, K., Auswanderungerscheinungen an schwingenden Kreiseln in kardanischer Lagerung, *Advances in Aeronautical Sciences*, Vol. 1, pp. 507-523. Pergamon Press, London and New York, 1957.

3. Lur'e, A.I., Zametki po analiticheskoi mekhanike (Notes on analytic mechanics). *PMM* Vol. 21, No. 6, 1957.

4. Pal'mgren, A., *Sharikovye i rolikovye podshipniki* (*Ball and Roller Bearings*). Mashgiz, 1949.

5. Klimov, D.M., O dvizhenii giroskopa v kardanovom podvese s neaksial'no nasazhennym rotorom (On the motion of a gyroscope on gimbals with a non-axially implaced rotor). *Dokl. Akad. Nauk SSSR* Vol.124, No. 3, 1959.

6. Yamamoto, T., On critical speeds of a shaft supported by a ball bearing. *J. Appl. Mech.* 26, No. 2, 1959.

Translated by V.C.